

# Yet another criterion for global existence in the 3D relativistic Vlasov-Maxwell system

MARKUS KUNZE

Mathematisches Institut, Universität Köln,  
Weyertal 86-90, D-50931 Köln, Germany  
e-mail: mkunze@mi.uni-koeln.de

## Abstract

We prove that solutions of the 3D relativistic Vlasov-Maxwell system can be extended, as long as the quantity  $\sigma_{-1}(t, x) = \max_{|\omega|=1} \int_{\mathbb{R}^3} \frac{dp}{\sqrt{1+p^2}} \frac{1}{(1+v \cdot \omega)} f(t, x, p)$  is bounded in  $L_x^2$ .

## 1 Introduction and main result

The relativistic Vlasov-Maxwell system describes the time evolution of a plasma, i.e., of an ensemble of charged particles (like ions or electrons) in position-momentum phase space  $\mathbb{R}^3 \times \mathbb{R}^3$ . Since the particles can move at relativistic speeds the motion of a single particle is described by the system

$$\dot{X} = V = \frac{P}{\sqrt{1+|P|^2}}, \quad \dot{V} = E + V \wedge B, \quad (1.1)$$

where we take only one species for simplicity and choose units where  $c = 1$  for the speed of light. Furthermore, the rest mass and the charge of the particles are set to unity. The particle velocity is  $V$ , whereas  $P$  denotes its momentum. The vectors  $E$  and  $B$  in (1.1) stand for the electric and the magnetic field, respectively. Since the number of individual particles in the plasma is large one takes a statistical approach and models the time evolution by using a density function  $f = f(t, x, p) \geq 0$  depending on time  $t \in \mathbb{R}$ , position  $x \in \mathbb{R}^3$ , and momentum  $p \in \mathbb{R}^3$ . Then the requirement that  $f$  be constant along the particle trajectories, i.e., the solutions of the characteristic equations (1.1), leads to the Vlasov equation

$$\partial_t f(t, x, p) + v \cdot \nabla f(t, x, p) + (E(t, x) + v \wedge B(t, x)) \cdot \nabla_p f(t, x, p) = 0; \quad (1.2)$$

here  $\nabla$  always means  $\nabla_x$ . The velocity  $v \in \mathbb{R}^3$  associated to  $p$  is

$$v = \frac{p}{\sqrt{1+p^2}}, \quad \text{thus} \quad p = \frac{v}{\sqrt{1-v^2}},$$

where  $p^2 = |p|^2$  and  $v^2 = |v|^2$  for brevity. The Lorentz force

$$L = L(t, x, v) = E(t, x) + v \wedge B(t, x) \in \mathbb{R}^3$$

is obtained from the fields  $E$  and  $B$ , which in turn satisfy the Maxwell equations

$$\partial_t E = \nabla \wedge B - j, \quad \nabla \cdot E = \rho, \quad (1.3)$$

and

$$\partial_t B = -\nabla \wedge E, \quad \nabla \cdot B = 0. \quad (1.4)$$

The coupling of (1.2) to (1.3), (1.4) is realized through the charge density  $\rho = \rho(t, x) \in \mathbb{R}$  and the current density  $j = j(t, x) \in \mathbb{R}^3$  via

$$\rho(t, x) = \int_{\mathbb{R}^3} f(t, x, p) dp \quad \text{and} \quad j(t, x) = \int_{\mathbb{R}^3} v f(t, x, p) dp. \quad (1.5)$$

Furthermore, initial data

$$f(t=0) = f^{(0)}, \quad E(t=0) = E^{(0)}, \quad \text{and} \quad B(t=0) = B^{(0)} \quad (1.6)$$

are prescribed such that the constraint equations

$$\nabla \cdot E^{(0)} = \rho^{(0)} = \int_{\mathbb{R}^3} f^{(0)} dp \quad \text{and} \quad \nabla \cdot B^{(0)} = 0 \quad (1.7)$$

are satisfied. Good general introductions to the subject can be found in [20, 3].

The relativistic Vlasov-Maxwell system comprises a complicated system of nonlinear partial differential equations. Local existence of solutions for smooth and compactly supported (or sufficiently decaying) initial data and a sufficient condition for global existence has been known for some time. More precisely, we have the following result.

**Theorem 1.1 (Glassey/Strauss)** *Let the initial data*

$$f^{(0)} \in C_0^1(\mathbb{R}^3 \times \mathbb{R}^3) \quad \text{and} \quad E^{(0)}, B^{(0)} \in C_b^2(\mathbb{R}^3; \mathbb{R}^3) \cap L^2(\mathbb{R}^3; \mathbb{R}^3)$$

*be given such that the constraint equations (1.7) are satisfied. Then there exists a maximal local solution  $(f, E, B)$  on a time interval  $[0, T_{\max}[$  to the relativistic Vlasov-Maxwell system (1.2), (1.3), (1.4), (1.5) such that the initial data are attained at  $t = 0$ ; see (1.6). Furthermore, if there is a function  $\varpi \in C([0, \infty[)$  so that*

$$P_\infty(t) \leq \varpi(t), \quad t \in [0, T_{\max}[, \quad (1.8)$$

*then  $T_{\max} = \infty$ .*

Here

$$P_\infty(t) = 10 + \sup \left\{ |p| : \exists s \in [0, t] \exists x \in \mathbb{R}^3 : f(s, x, p) \neq 0 \right\}$$

is the maximal momentum up to the time  $t$ . For a proof, see [7], or [11, 1] for the same result obtained by different methods. The work [7] has been generalized to initial data of non-compact support in [9, 12], the latter is also extending [10, 16]. The problem of unrestricted global existence has been studied by many people. Only in the framework of weak solution it has been solved in [2]. Regarding classical solutions, adding spherical symmetry [10], smallness of initial data [8], or “near neutrality” [4, 16] has turned out to be sufficient to close the case. Another remarkable

work is [5], where global existence was shown for the “two and one-half-dimensional” system, i.e.,  $x \in \mathbb{R}^2$  and  $p \in \mathbb{R}^3$ .

Due to the lack of a general global result in 3D it is natural to focus on deriving further continuation criteria, apart from (1.8), which might be easier to check (although, of course, all criteria would be equivalent in the end if global existence was known). The following result summarizes some early attempts in this direction.

**Theorem 1.2** *The following are equivalent:*

(a)  $T_{\max} = \infty$ .

(b) *There is a function  $\varpi_1 \in C([0, \infty[)$  such that  $P_\infty(t) \leq \varpi_1(t)$  for  $t \in [0, T_{\max}[$ .*

(c) *There is a function  $\varpi_2 \in C([0, \infty[)$  such that*

$$\sup \left\{ \int_{\mathbb{R}^3} \sqrt{1 + p^2} f(t, x, p) dp : x \in \mathbb{R}^3 \right\} \leq \varpi_2(t), \quad t \in [0, T_{\max}[.$$

(d) *There is a function  $\varpi_3 \in C([0, \infty[)$  such that*

$$\sup \left\{ |P(t; 0, x, p) - p| : (x, p) \in \mathbb{R}^3 \times \mathbb{R}^3 \right\} \leq \varpi_3(t), \quad t \in [0, T_{\max}[.$$

*Here  $s \mapsto (X(s; t, x, p), P(s; t, x, p))$  is the solution to the characteristic system (1.1) which at  $s = t$  equals  $(x, p)$ .*

(e) (i)  $f^{(0)} = 0$  or

(ii) *there is  $\varepsilon > 0$  and  $R_0 > 0$  such that  $\int_{|x| \leq R_0} \int_{\mathbb{R}^3} f^{(0)}(x, p) dx dp > \varepsilon$  so that for all  $R \in [0, R_0 + T_{\max}[$  it holds that*

$$\lim_{t \rightarrow T_{\max}} \int_{|x| \leq R} \left[ \int_{\mathbb{R}^3} \sqrt{1 + p^2} f(t, x, p) dp + \frac{1}{2} (|E(t, x)|^2 + |B(t, x)|^2) \right] dx = 0.$$

Parts (c) and (e) are more or less contained in [9] and [6], respectively; see [12] for a detailed proof. After some dormant period recently the interest in the subject has been revived and further criteria have been obtained, using refined techniques. To state the results we need to introduce the quantity

$$\mathcal{I}_\theta(t, x) = \int_{\mathbb{R}^3} (1 + p^2)^{\frac{\theta}{2}} f(t, x, p) dp \tag{1.9}$$

for  $\theta > 0$ .

**Theorem 1.3** *The following are equivalent:*

(a)  $T_{\max} = \infty$ .

(f) *There is a function  $\varpi_4 \in C([0, \infty[)$  such that*

$$\sup \left\{ |\mathbb{P}_Q p| : \exists s \in [0, t] \exists x \in \mathbb{R}^3 : f(s, x, p) \neq 0 \right\} \leq \varpi_4(t), \quad t \in [0, T_{\max}[.$$

*Here  $Q \subset \mathbb{R}^3$  is a two-dimensional plane in  $p$ -space containing the origin and  $\mathbb{P}$  denotes the projection onto  $Q$ .*

(g) There is a function  $\varpi_5 \in C([0, \infty[)$  such that

$$\|\rho(t)\|_{L_x^\infty(\mathbb{R}^3)} = \|\mathcal{I}_0(t)\|_{L_x^\infty(\mathbb{R}^3)} \leq \varpi_5(t), \quad t \in [0, T_{\max}[.$$

(h) Let  $q \in ]2, \infty]$  and  $\theta > 2/q$ , or  $q \in [1, 2]$  and  $\theta > 8/q - 3$ . Then there is a function  $\varpi_6 \in C([0, \infty[)$  such that

$$\|\mathcal{I}_\theta(t)\|_{L_x^q(\mathbb{R}^3)} \leq \varpi_6(t), \quad t \in [0, T_{\max}[.$$

Part (f), in fact for more general data, is due to [13]. It shows that one does not have to control the full momentum support, but only its projection to some plane through the origin. Part (g) has been proved in [19]. The most recent result is (h), which is cited from [14], and once again holds for more general data. The results from [14] generalize those of [15], where  $q \in [6, \infty[$  and  $\theta > 4/q$ , or  $q < 6$  and  $\theta > 22/q - 3$  was assumed.

In this work we propose to study another quantity, which is

$$\sigma_{-1}(t, x) = \max_{|\omega|=1} \int_{\mathbb{R}^3} \frac{dp}{\sqrt{1+p^2}} \frac{1}{(1+v \cdot \omega)} f(t, x, p) \quad (1.10)$$

and which comes up naturally in the course of the estimates. Our main result is as follows.

**Theorem 1.4** *Suppose that the initial data*

$$f^{(0)} \in C_0^1(\mathbb{R}^3 \times \mathbb{R}^3) \quad \text{and} \quad E^{(0)}, B^{(0)} \in C_b^2(\mathbb{R}^3; \mathbb{R}^3) \cap L^2(\mathbb{R}^3; \mathbb{R}^3)$$

*are given such that the constraint equations (1.7) are satisfied. Let  $\sigma_{-1}$  be defined by (1.10). If there is a function  $\varpi \in C([0, \infty[)$  so that*

$$\|\sigma_{-1}\|_{L_t^\infty L_x^2(S_T)} \leq \varpi(T), \quad T \in [0, T_{\max}[ , \quad (1.11)$$

*for  $S_T = [0, T] \times \mathbb{R}^3$ , then  $T_{\max} = \infty$ .*

The point is that  $\int_{\mathbb{R}^3} \frac{dp}{\sqrt{1+p^2}} f \in L_t^\infty L_x^2$  by energy conservation, so as compared to  $\sigma_{-1}$  one might hope to “get away with a logarithmic loss”. The method of proof is somewhat similar to [14], in that Strichartz estimates for a wave equation related to  $E$  and  $B$  are applied. However, we can avoid the use of iteration sequences and bounds on the field derivatives, which makes the argument more direct. Comparing  $\|\sigma_{-1}(t)\|_{L_x^2(\mathbb{R}^3)}$  to  $\|\mathcal{I}_\theta(t)\|_{L_x^q(\mathbb{R}^3)}$  one can also derive a corollary in the fashion of Theorem 1.3(h).

**Corollary 1.5** *Under the hypotheses of Theorem 1.4, let  $\theta > 1$  and  $q \in ]\frac{4}{\theta+1}, \infty[$  be given. Then  $T_{\max} = \infty$  is equivalent to the existence of a function  $\varpi_7 \in C([0, \infty[)$  such that*

$$\|\mathcal{I}_\theta(t)\|_{L_x^q(\mathbb{R}^3)} \leq \varpi_7(t), \quad t \in [0, T_{\max}[.$$

*In particular, this gives something new as compared to Theorem 1.3(h) for  $\theta > 1$  and  $q \in ]\frac{4}{\theta+1}, \frac{8}{\theta+3}]$ .*

The paper is organized as follows. In Section 2 we collect some preliminary results which are well-known in general. Then we turn to deriving suitable bounds on  $E$  and  $B$  in Section 3; they mainly rely on the representation formulae for the fields due to Glassey and Strauss and on Strichartz estimates for the wave equation. The argument for the proof of Theorem 1.4 is elaborated in Section 4. Finally, Section 5 contains the proof of Corollary 1.5.

Constants which only depend on the initial data are denoted by  $C(0)$ , whereas  $C$  is a numerical constant. Sometimes data terms are not made explicit and are only written as “(data)”. By our hypotheses they are good enough with regard to all the estimates we will be aiming for.

The wave operator on  $\mathbb{R} \times \mathbb{R}^3$  is  $\square = \partial_t^2 - \Delta$ . For functions  $h = h(t, x)$  define

$$\begin{aligned} (\square^{-1}h)(t, x) &= \int_0^t \frac{ds}{4\pi s} \int_{|y-x|=s} dS(y) h(t-s, y) = \int_{|y-x| \leq t} \frac{dy}{4\pi|x-y|} h(t-|x-y|, y) \\ &= \int_{|y| \leq t} \frac{dy}{4\pi|y|} h(t-|y|, x+y). \end{aligned} \quad (1.12)$$

Then  $g = \square^{-1}h$  is the unique solution to

$$\square g = h, \quad g(0) = \partial_t g(0) = 0.$$

## 2 Some preliminaries

From the system (1.2), (1.3), and (1.4) it follows that the energy

$$\mathcal{E}(t) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \sqrt{1+p^2} f(t, x, p) dx dp + \frac{1}{2} \int_{\mathbb{R}^3} (|E(t, x)|^2 + |B(t, x)|^2) dx$$

is conserved along sufficiently regular solutions (as we are dealing with); note that  $\nabla_p \sqrt{1+p^2} = v$ . In addition, (1.5) and (1.2) yield the continuity equation  $\partial_t \rho + \nabla \cdot j = 0$ .

For  $(t, x, p)$  fixed let  $s \mapsto (X(s; t, x, p), P(s; t, x, p))$  denote the solution of the characteristic initial value problem (1.1) which at  $s = t$  equals  $(x, p)$ . Then (1.2) is equivalent to

$$\frac{d}{ds} f(s, X(s, t, x, p), P(s, t, x, p)) = 0,$$

which leads to the relation

$$f(t, x, p) = f^{(0)}(X(0, t, x, p), P(0, t, x, p)) \quad (2.1)$$

for the initial data  $f^{(0)}(x, p) = f(0, x, p)$ . Thus in particular

$$\mathcal{L}(t) = \|f(t)\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} = \|f^{(0)}\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} = \mathcal{L}(0).$$

Since every map  $(x, p) \mapsto (X(s; t, x, p), P(s; t, x, p))$  is a measure preserving diffeomorphism of  $\mathbb{R}^3 \times \mathbb{R}^3$ , it follows from (2.1) that for instance

$$\|\rho(t)\|_{L^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f(t, x, p) dx dp = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f^{(0)}(\bar{x}, \bar{p}) d\bar{x} d\bar{p} = \|\rho^{(0)}\|_{L^1(\mathbb{R}^3)},$$

where  $\rho^{(0)}(x) = \int_{\mathbb{R}^3} f^{(0)}(x, p) dp$ , which expresses the conservation of mass.

The following result is also known, but we nevertheless include a proof in order to make the presentation self-contained.

**Lemma 2.1** *Let  $m_k$  be defined by*

$$m_k(t) = 1 + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + p^2)^{\frac{k}{2}} f(t, x, p) dx dp.$$

*If  $k \in [2, \infty[$ , then*

$$m_k(t) \leq m_k(0) + C\mathcal{L}(0)^{\frac{1}{k+3}} \int_0^t \|E(s)\|_{L_x^{k+3}(\mathbb{R}^3)} m_k(s)^{\frac{k+2}{k+3}} ds,$$

*where  $C$  depends on  $k$ .*

**Proof:** The Vlasov equation (1.2) yields

$$\begin{aligned} \frac{dm_k}{dt} &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + p^2)^{\frac{k}{2}} \partial_t f dx dp = - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + p^2)^{\frac{k}{2}} \nabla_p \cdot ((E + v \wedge B)f) dx dp \\ &= k \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + p^2)^{\frac{k}{2}-1} p \cdot ((E + v \wedge B)f) dx dp \\ &= k \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + p^2)^{\frac{k-1}{2}} (v \cdot E) f dx dp, \end{aligned}$$

so that

$$\left| \frac{dm_k}{dt} \right| \leq k \|E\|_{L_x^{q'}(\mathbb{R}^3)} \left\| \int_{\mathbb{R}^3} (1 + p^2)^{\frac{k-1}{2}} f dp \right\|_{L_x^q(\mathbb{R}^3)} \quad (2.2)$$

for  $q \in [1, \infty]$ . If  $R \in ]0, \infty[$  and  $\theta \in ]0, \infty[$  are fixed, then

$$\begin{aligned} \int_{\mathbb{R}^3} (1 + p^2)^{\frac{k-1}{2}} f dp &= \int_{|p| \leq R} (1 + p^2)^{\frac{k-1}{2}} f dp + \int_{|p| > R} (1 + p^2)^{\frac{k-1}{2}} f dp \\ &\leq \mathcal{L}(0) \int_{|p| \leq R} (1 + p^2)^{\frac{k-1}{2}} dp + R^{-2\theta} \int_{|p| > R} (1 + p^2)^{\frac{k-1}{2} + \theta} f dp \\ &\leq C\mathcal{L}(0)(1 + R)^{k+2} + R^{-2\theta} \int_{\mathbb{R}^3} (1 + p^2)^{\frac{k-1}{2} + \theta} f dp, \end{aligned}$$

where  $C$  depends on  $k$ . Tacitly assuming  $R \in [1, \infty[$  for the optimal  $R$  (otherwise the compact  $x$ -support is useful to obtain the needed bound), this yields

$$\int_{\mathbb{R}^3} (1 + p^2)^{\frac{k-1}{2}} f dp \leq C\mathcal{L}(0)^{\frac{2\theta}{k+2(1+\theta)}} \left( \int_{\mathbb{R}^3} (1 + p^2)^{\frac{k-1}{2} + \theta} f dp \right)^{\frac{k+2}{k+2(1+\theta)}},$$

where  $C$  depends on  $k$  and  $\theta$ . Therefore the estimate

$$\left\| \int_{\mathbb{R}^3} (1 + p^2)^{\frac{k-1}{2}} f dp \right\|_{L_x^{\frac{k+2(1+\theta)}{k+2}}(\mathbb{R}^3)} \leq C\mathcal{L}(0)^{\frac{2\theta}{k+2(1+\theta)}} m_{k-1+2\theta}^{\frac{k+2}{k+2(1+\theta)}}$$

is found. Putting  $q = 1 + \frac{2\theta}{k+2}$ , from  $k - 1 + 2\theta = (k + 2)q - 3$  it follows that

$$\left\| \int_{\mathbb{R}^3} (1 + p^2)^{\frac{k-1}{2}} f dp \right\|_{L_x^q(\mathbb{R}^3)} \leq C\mathcal{L}(0)^{\frac{1}{q}} m_{(k+2)q-3}^{\frac{1}{q}}$$

is verified for  $k \in [2, \infty[$  and  $q \in ]1, \infty[$ , where  $C$  depends on  $k$  and  $q$ . Using this in (2.2),

$$\left| \frac{dm_k}{dt} \right| \leq C\mathcal{L}(0)^{\frac{1}{q}} \|E\|_{L_x^{q'}(\mathbb{R}^3)} m_{(k+2)q-3}^{\frac{1}{q}}.$$

For the particular choice of  $q = \frac{k+3}{k+2}$  and  $q' = k + 3$ , the claimed bound is obtained.  $\square$

**Lemma 2.2** *If  $q \in ]1, \infty[$  and  $\alpha \in [0, 1 - \frac{1}{2q}]$ , then*

$$\|\sigma_{-1}(t)\|_{L^q_x(\mathbb{R}^3)} \leq C P_\infty(t)^{2(1-\alpha)-\frac{1}{q}} m_{2\alpha q}(t)^{\frac{1}{q}},$$

where  $C$  depends on  $q$ ,  $\alpha$ , and  $\mathcal{L}(0)$ .

**Proof:** Fix  $\omega \in \mathbb{R}^3$  such that  $|\omega| = 1$ . Then by Hölder's inequality and by Lemma 2.3(b) below for  $\theta = (\alpha + \frac{1}{2})q'$  and  $\kappa = q'$ ,

$$\begin{aligned} & \int_{\mathbb{R}^3} \frac{dp}{\sqrt{1+p^2}} \frac{1}{(1+v \cdot \omega)} f(t) \\ &= \int_{\mathbb{R}^3} \frac{dp}{(1+p^2)^{\alpha+\frac{1}{2}}} \frac{1}{(1+v \cdot \omega)} (1+p^2)^\alpha f(t) \\ &\leq \left( \int_{|p| \leq P_\infty(t)} \frac{dp}{(1+p^2)^{(\alpha+\frac{1}{2})q'}} \frac{1}{(1+v \cdot \omega)^{q'}} \right)^{1/q'} \left( \int_{\mathbb{R}^3} (1+p^2)^{\alpha q} f(t)^q dp \right)^{1/q} \\ &\leq C(\alpha, q) \mathcal{L}(0)^{\frac{q-1}{q}} P_\infty(t)^{2(1-\alpha)-\frac{1}{q}} \left( \int_{\mathbb{R}^3} (1+p^2)^{\alpha q} f(t) dp \right)^{1/q}. \end{aligned}$$

Taking first the  $\max_{|\omega|=1}$ , then the  $q'$ th power, and then integrating  $\int_{\mathbb{R}^3} dx$ , the claimed bound is obtained.  $\square$

The following general integration lemma is useful at many places.

**Lemma 2.3** *Suppose that  $R \in [10, \infty[$  and  $|\omega| = 1$ .*

(a) *If  $\theta \in [0, \frac{3}{2}]$ , then*

$$\int_{|p| \leq R} \frac{dp}{(1+p^2)^\theta} \frac{1}{1+v \cdot \omega} \leq C \ln R R^{3-2\theta},$$

where  $C$  depends on  $\theta$ .

(b) *If  $\theta, \kappa \in [0, \infty[$  are such that  $\theta < \kappa + \frac{1}{2}$  and  $\kappa > 1$ , then*

$$\int_{|p| \leq R} \frac{dp}{(1+p^2)^\theta} \frac{1}{(1+v \cdot \omega)^\kappa} \leq C R^{1+2(\kappa-\theta)},$$

where  $C$  depends on  $\theta$  and  $\kappa$ .

(c) *If  $\theta \in [0, 1[$  and  $|v| < 1$ , then*

$$\int_{|\omega|=1} \frac{dS(\omega)}{(1+v \cdot \omega)^\theta} \leq C,$$

where  $C$  depends on  $\theta$ .

**Proof:** (a) First  $\omega$  is rotated to  $(0, 0, 1)$ . Then spherical coordinates and the transformation

$$\sigma = \frac{r}{\sqrt{1+r^2}}, \quad r = \frac{\sigma}{\sqrt{1-\sigma^2}}, \quad d\sigma = (1-\sigma^2)^{3/2} dr, \quad 1+r^2 = (1-\sigma^2)^{-1}, \quad (2.3)$$

are used to get

$$\begin{aligned}
\int_{|p| \leq R} \frac{dp}{(1+p^2)^\theta} \frac{1}{1+v \cdot \omega} &= \int_{|p| \leq R} \frac{dp}{(1+p^2)^\theta} \frac{1}{1+v_3} \leq C \int_0^R \frac{dr r^2}{(1+r^2)^\theta} \frac{1}{1+\frac{r \cos \varphi}{\sqrt{1+r^2}}} \\
&= C \int_0^{R^b} \frac{d\sigma \sigma^2}{(1-\sigma^2)^{\frac{5}{2}-\theta}} \int_{-1}^1 \frac{ds}{1+\sigma s} \\
&= C \int_0^{R^b} \frac{d\sigma \sigma}{(1-\sigma^2)^{\frac{5}{2}-\theta}} \ln \left( \frac{1+\sigma}{1-\sigma} \right).
\end{aligned}$$

Since  $\ln(\frac{1+\sigma}{1-\sigma}) \leq \ln(\frac{4}{1-\sigma^2}) \leq \ln(\frac{4}{1-(R^b)^2}) = \ln(4(1+R^2)) \leq \ln(8R^2) \leq 3 \ln R$ , the claim follows. (b) Similar as in (a),

$$\begin{aligned}
\int_{|p| \leq R} \frac{dp}{(1+p^2)^\theta} \frac{1}{(1+v \cdot \omega)^\kappa} &\leq C \int_0^{R^b} \frac{d\sigma \sigma^2}{(1-\sigma^2)^{\frac{5}{2}-\theta}} \int_{-1}^1 \frac{ds}{(1+\sigma s)^\kappa} \\
&\leq C \int_0^{R^b} \frac{d\sigma \sigma}{(1-\sigma^2)^{\frac{5}{2}-\theta}} \frac{1}{(1-\sigma)^{\kappa-1}} \\
&\leq C \int_0^{R^b} \frac{d\sigma \sigma}{(1-\sigma^2)^{\frac{3}{2}-\theta+\kappa}} \leq C R^{1+2(\kappa-\theta)}.
\end{aligned}$$

(c) First consider the case where  $|v| \leq 1/2$ . Then  $1+v \cdot \omega \geq 1-|v| \geq 1/2$  yields

$$\int_{|\omega|=1} \frac{dS(\omega)}{(1+v \cdot \omega)^\theta} \leq 4\pi 2^\theta.$$

If  $|v| \geq 1/2$ , then  $v$  is rotated to  $(0, 0, |v|)$  to get

$$\begin{aligned}
\int_{|\omega|=1} \frac{dS(\omega)}{(1+v \cdot \omega)^\theta} &= \int_{|\omega|=1} \frac{dS(\omega)}{(1+|v|\omega_3)^\theta} = \int_0^{2\pi} d\varphi \int_0^\pi d\theta \frac{\sin \theta}{(1+\cos \theta |v|)^\theta} \\
&= 2\pi \int_{-1}^1 \frac{ds}{(1+s|v|)^\theta} = \frac{2\pi}{(1-\theta)|v|} \left( (1+|v|)^{1-\theta} - (1-|v|)^{1-\theta} \right) \\
&\leq \frac{4\pi 2^{1-\theta}}{(1-\theta)}.
\end{aligned}$$

This completes the proof. □

### 3 Bounds on the fields

First we recall the following representation of the fields  $E$  and  $B$  from [17, (A13), (A14), (A3)].

$$E = E_D + E_{DT} + E_b + E_\sharp, \quad (3.4)$$

$$B = B_D + B_{DT} + B_b + B_\sharp, \quad (3.5)$$



where

$$\begin{aligned}
E_D(t, x) &= \partial_t \left( \frac{t}{4\pi} \int_{|\omega|=1} E^{(0)}(x + t\omega) d\omega \right) + \frac{t}{4\pi} \int_{|\omega|=1} \partial_t E(0, x + t\omega) d\omega \quad (\text{data}), \\
E_{DT}(t, x) &= -\frac{1}{t} \int_{|y|=t} \int_{\mathbb{R}^3} K_{E,DT}(\omega, v) f^{(0)}(x + y, p) dp d\sigma(y) \quad (\text{data}), \\
E_b(t, x) &= -\int_{|y|\leq t} \frac{dy}{|y|^2} \int_{\mathbb{R}^3} dp K_{E,b}(\omega, v) f(t - |y|, x + y, p), \\
E_{\#}(t, x) &= -\int_{|y|\leq t} \frac{dy}{|y|} \int_{\mathbb{R}^3} dp K_{E,\#}(\omega, v) (Lf)(t - |y|, x + y, p),
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
B_D(t, x) &= \partial_t \left( \frac{t}{4\pi} \int_{|\omega|=1} B^{(0)}(x + t\omega) d\omega \right) + \frac{t}{4\pi} \int_{|\omega|=1} \partial_t B(0, x + t\omega) d\omega \quad (\text{data}), \\
B_{DT}(t, x) &= \frac{1}{t} \int_{|y|=t} \int_{\mathbb{R}^3} K_{B,DT}(\omega, v) f^{(0)}(x + y, p) dp d\sigma(y) \quad (\text{data}), \\
B_b(t, x) &= \int_{|y|\leq t} \frac{dy}{|y|^2} \int_{\mathbb{R}^3} dp K_{B,b}(\omega, v) f(t - |y|, x + y, p), \\
B_{\#}(t, x) &= \int_{|y|\leq t} \frac{dy}{|y|} \int_{\mathbb{R}^3} dp K_{B,\#}(\omega, v) (Lf)(t - |y|, x + y, p),
\end{aligned}$$

defining  $\omega = |y|^{-1}y$ . The respective kernels are given by

$$\begin{aligned}
K_{E,DT}(\omega, v) &= (1 + v \cdot \omega)^{-1}(\omega - (v \cdot \omega)v), \\
K_{E,b}(\omega, v) &= (1 + p^2)^{-1}(1 + v \cdot \omega)^{-2}(v + \omega), \\
K_{E,\#}(\omega, v) &= (1 + p^2)^{-1/2}(1 + v \cdot \omega)^{-2} \left[ (1 + v \cdot \omega) + ((v \cdot \omega)\omega - v) \otimes v \right. \\
&\quad \left. - (v + \omega) \otimes \omega \right] \in \mathbb{R}^{3 \times 3},
\end{aligned}$$

and

$$\begin{aligned}
K_{B,DT}(\omega, v) &= -(1 + v \cdot \omega)^{-1}(v \wedge \omega), \\
K_{B,b}(\omega, v) &= -(1 + p^2)^{-1}(1 + v \cdot \omega)^{-2}(v \wedge \omega), \\
K_{B,\#}(\omega, v) &= (1 + p^2)^{-1/2}(1 + v \cdot \omega)^{-2} \left[ (1 + v \cdot \omega) \omega \wedge (\dots) \right. \\
&\quad \left. - (v \wedge \omega) \otimes (v + \omega) \right] \in \mathbb{R}^{3 \times 3}.
\end{aligned}$$

**Lemma 3.1** *The following (known) estimates hold.*

$$\begin{aligned}
|K_{E,DT}(\omega, v)| + |K_{B,DT}(\omega, v)| &\leq C(1 + v \cdot \omega)^{-1/2}, \\
|K_{E,b}(\omega, v)| + |K_{B,b}(\omega, v)| &\leq C(1 + p^2)^{-1}(1 + v \cdot \omega)^{-3/2}, \\
|K_{E,\#}(\omega, v)z| + |K_{B,\#}(\omega, v)z| &\leq C(1 + p^2)^{-1/2}(1 + v \cdot \omega)^{-1}|z| \quad (z \in \mathbb{R}^3).
\end{aligned} \tag{3.7}$$

**Proof:** The first two lines are a consequence of

$$\begin{aligned}
|\omega - (v \cdot \omega)v| &= \left(1 - 2(v \cdot \omega)^2 + (v \cdot \omega)^2 v^2\right)^{1/2} \leq \left(1 - (v \cdot \omega)^2\right)^{1/2} \leq \sqrt{2}(1 + v \cdot \omega)^{1/2}, \\
|v + \omega| &= \left(v^2 + 2(v \cdot \omega) + 1\right)^{1/2} \leq \sqrt{2}(1 + v \cdot \omega)^{1/2}, \\
|v \wedge \omega| &= |(v + \omega) \wedge \omega| \leq |v + \omega| \leq \sqrt{2}(1 + v \cdot \omega)^{1/2}.
\end{aligned} \tag{3.8}$$

The bound on  $|K_{B,\sharp}(\omega, v)z|$  is immediate from the preceding estimates. To bound  $|K_{E,\sharp}(\omega, v)z|$ , finally note that

$$\begin{aligned}
\left[ ((v \cdot \omega)\omega - v) \otimes v - (v + \omega) \otimes \omega \right] z &= (v \cdot z)((v \cdot \omega)\omega - v) - (\omega \cdot z)(v + \omega) \\
&= -(\omega - (v \cdot \omega)v) \cdot z(v + \omega) - (1 + v \cdot \omega)(v \cdot z)v.
\end{aligned}$$

This yields the claim.  $\square$

For functions  $h = h(t, x)$  define the operator  $\mathcal{W}$  by

$$\begin{aligned}
(\mathcal{W}h)(t, x) &= \int_0^t \frac{ds}{4\pi s^2} \int_{|y-x|=s} dS(y) h(t-s, y) = \int_{|y-x| \leq t} \frac{dy}{4\pi|x-y|^2} h(t-|x-y|, y) \\
&= \int_{|y| \leq t} \frac{dy}{4\pi|y|^2} h(t-|y|, x+y).
\end{aligned} \tag{3.9}$$

**Lemma 3.2** *The following estimates hold.*

$$|E_D(t, x)| + |E_{DT}(t, x)| + |B_D(t, x)| + |B_{DT}(t, x)| \leq C(\text{data}), \tag{3.10}$$

$$|E_b(t, x)| + |B_b(t, x)| \leq C(\mathcal{W}\sigma_{-1})(t, x), \tag{3.11}$$

$$|E_{\sharp}(t, x)| + |B_{\sharp}(t, x)| \leq C\left(\square^{-1}(|E| + |B|)\sigma_{-1}\right)(t, x). \tag{3.12}$$

**Proof:** Concerning the second pair of estimates, by Lemma 3.1 for instance

$$\begin{aligned}
|E_b(t, x)| &\leq C \int_{|y| \leq t} \frac{dy}{|y|^2} \int \frac{dp}{1+p^2} \frac{1}{(1+v \cdot \omega)^{3/2}} f(t-|y|, x+y, p) \\
&\leq C \int_{|y| \leq t} \frac{dy}{|y|^2} \int \frac{dp}{\sqrt{1+p^2}} \frac{1}{(1+v \cdot \omega)} f(t-|y|, x+y, p) \\
&\leq C \int_{|y| \leq t} \frac{dy}{|y|^2} \sigma_{-1}(t-|y|, x+y) = C(\mathcal{W}\sigma_{-1})(t, x),
\end{aligned}$$

using the trivial bound  $1 + v \cdot \omega \geq 1 - |v| \geq \frac{1}{2}(1 - v^2) = \frac{1}{2(1+p^2)}$ , so that  $(1 + v \cdot \omega)^{-1/2} \leq \sqrt{2} \sqrt{1+p^2}$ . The same argument can be used to show that also  $|B_b(t, x)| \leq C(\mathcal{W}\sigma_{-1})(t, x)$ . For  $E_{\sharp}$ , again Lemma 3.1 may be invoked to give

$$\begin{aligned}
|E_{\sharp}(t, x)| &\leq C \int_{|y| \leq t} \frac{dy}{|y|^2} \int \frac{dp}{\sqrt{1+p^2}} \frac{1}{(1+v \cdot \omega)} (|L|f)(t-|y|, x+y, p) \\
&\leq C \int_{|y| \leq t} \frac{dy}{|y|^2} (|E| + |B|)(t-|y|, x+y) \sigma_{-1}(t-|y|, x+y) \\
&= C\left(\square^{-1}(|E| + |B|)\sigma_{-1}\right)(t, x),
\end{aligned}$$

recall (1.12). The bound on  $|B_{\sharp}(t, x)|$  is analogous.  $\square$

For the wave equation the following Strichartz estimates are known; see [18, (4.9), p. 100]. For every  $\gamma \in ]0, 1[$  there is a constant  $C_{\gamma}^* > 0$  with the following property. Let  $u$  be a solution to  $\square u = F$  on a strip  $[a, b] \times \mathbb{R}^3$ .

$$\begin{aligned} & \|u\|_{L_t^{\frac{2}{\gamma}} L_x^{\frac{2}{1-\gamma}}([a, b] \times \mathbb{R}^3)} + \|u\|_{C_t \dot{H}_x^{\gamma}([a, b] \times \mathbb{R}^3)} + \|\partial_t u\|_{C_t \dot{H}_x^{\gamma-1}([a, b] \times \mathbb{R}^3)} \\ & \leq C_{\gamma}^* \left( \|u(a)\|_{\dot{H}_x^{\gamma}(\mathbb{R}^3)} + \|\partial_t u(a)\|_{\dot{H}_x^{\gamma-1}(\mathbb{R}^3)} + \|F\|_{L_t^{\frac{2}{1+\gamma}} L_x^{\frac{2}{2-\gamma}}([a, b] \times \mathbb{R}^3)} \right). \end{aligned} \quad (3.13)$$

The constant  $C_{\gamma}^*$  is independent of  $a$  and  $b$ .

Next an estimate for  $\mathcal{W}$  is derived. It might not be optimal, but it will turn out to be sufficient in the sequel.

**Lemma 3.3** *Let  $\mathcal{W}$  be defined by (3.9). If  $T > 0$  and  $S_T = [0, T] \times \mathbb{R}^3$ , then*

$$\|\mathcal{W}h\|_{L_t^{\infty} \dot{H}_x^{1-\varepsilon}(S_T)} \leq C \varepsilon^{-1} T^{\varepsilon} \|h\|_{L_t^{\infty} L_x^2(S_T)}, \quad \varepsilon \in ]0, 1]. \quad (3.14)$$

In particular,

$$\|\mathcal{W}h\|_{L_t^{\infty} L_x^{\frac{6}{1+2\varepsilon}}(S_T)} \leq C_2(\varepsilon, T) \|h\|_{L_t^{\infty} L_x^2(S_T)}, \quad (3.15)$$

where  $C_2(\varepsilon, T) = C_1(\varepsilon) C \varepsilon^{-1} T^{\varepsilon}$  is increasing in  $T$ .

**Proof:** The Fourier transform of  $\mathcal{W}h$  is

$$\begin{aligned} \widehat{(\mathcal{W}h)}(t, \xi) &= \int_{\mathbb{R}^3} e^{-i\xi \cdot x} (\mathcal{W}h)(t, x) dx = \int_{|y| \leq t} \frac{dy}{4\pi|y|^2} \int_{\mathbb{R}^3} dx e^{-i\xi \cdot x} h(t - |y|, x + y) \\ &= \int_{|y| \leq t} \frac{dy}{4\pi|y|^2} e^{i\xi \cdot y} \hat{h}(t - |y|, \xi) = \int_0^t \frac{ds}{4\pi} \hat{h}(t - s, \xi) \int_{|\omega|=1} dS(\omega) e^{is\xi \cdot \omega} \\ &= \int_0^t \frac{\sin(s|\xi|)}{s|\xi|} \hat{h}(t - s, \xi) ds. \end{aligned}$$

Now use  $|\sin(s|\xi|)| \leq \min\{1, s|\xi|\}$  to obtain for  $\varepsilon \in ]0, 1[$

$$\begin{aligned} |\widehat{(\mathcal{W}h)}(t, \xi)| &\leq \int_0^t \mathbf{1}_{\{1 \leq s|\xi|\}} \frac{1}{s|\xi|} |\hat{h}(t - s, \xi)| ds + \int_0^t \mathbf{1}_{\{1 < \frac{1}{s|\xi|}\}} |\hat{h}(t - s, \xi)| ds \\ &\leq \int_0^t \mathbf{1}_{\{1 \leq s|\xi|\}} \frac{1}{s|\xi|} (s|\xi|)^{\varepsilon} |\hat{h}(t - s, \xi)| ds + \int_0^t \mathbf{1}_{\{1 < \frac{1}{s|\xi|}\}} \left(\frac{1}{s|\xi|}\right)^{1-\varepsilon} |\hat{h}(t - s, \xi)| ds \\ &\leq \frac{2}{|\xi|^{1-\varepsilon}} \int_0^t \frac{ds}{s^{1-\varepsilon}} |\hat{h}(t - s, \xi)|. \end{aligned}$$

It follows that

$$\begin{aligned} \|(\mathcal{W}h)(t)\|_{\dot{H}_x^{1-\varepsilon}(\mathbb{R}^3)} &= \frac{1}{(2\pi)^{3/2}} \| |\xi|^{1-\varepsilon} \widehat{(\mathcal{W}h)}(t) \|_{L_{\xi}^2(\mathbb{R}^3)} \\ &\leq 2 \int_0^t \frac{ds}{s^{1-\varepsilon}} \|h(t - s)\|_{L_x^2(\mathbb{R}^3)}, \end{aligned}$$

and this yields (3.14). By the homogeneous Sobolev embedding in  $\mathbb{R}^3$ ,

$$\|\mathcal{W}h\|_{L_t^\infty L_x^{\frac{6}{1+2\varepsilon}}(S_T)} \leq C_1(\varepsilon) \|\mathcal{W}h\|_{L_t^\infty \dot{H}_x^{1-\varepsilon}(S_T)} \leq C_1(\varepsilon) C \varepsilon^{-1} T^\varepsilon \|h\|_{L_t^\infty L_x^2(S_T)},$$

which is (3.15).  $\square$

**Corollary 3.4** *For  $\varepsilon \in ]0, 1]$ ,*

$$\left\| |E_b| + |B_b| \right\|_{L_t^\infty L_x^{\frac{6}{1+2\varepsilon}}(S_T)} \leq C_3(\varepsilon, T) \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_T)},$$

where  $C_3$  is increasing in  $T$ . In particular,

$$\left\| \square^{-1}((|E_b| + |B_b|)\sigma_{-1}) \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}(S_T)} \leq C_4(\varepsilon, T) \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_T)}^2 \quad (3.16)$$

for a constant  $C_4$  that is increasing in  $T$ .

**Proof:** The first bound is an immediate consequence of (3.11) and (3.15). To prove (3.16), note that

$$\begin{aligned} \left\| (|E_b| + |B_b|)\sigma_{-1} \right\|_{L_t^\infty L_x^{\frac{3}{2+\varepsilon}}(S_T)} &\leq \left\| |E_b| + |B_b| \right\|_{L_t^\infty L_x^{\frac{6}{1+2\varepsilon}}(\mathbb{R}^3)} \|\sigma_{-1}\|_{L_t^\infty L_x^2(\mathbb{R}^3)} \\ &\leq C_3(\varepsilon, T) \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_T)}^2. \end{aligned}$$

Thus using (3.13) for  $\gamma_\varepsilon = \frac{2}{3}(1 - \varepsilon)$ , where  $\frac{2}{2-\gamma_\varepsilon} = \frac{3}{2+\varepsilon}$ ,  $\frac{2}{1+\gamma_\varepsilon} = \frac{6}{5-2\varepsilon}$ ,  $\frac{2}{1-\gamma_\varepsilon} = \frac{6}{1+2\varepsilon}$ , and  $\frac{2}{\gamma_\varepsilon} = \frac{3}{1-\varepsilon}$ ,

$$\begin{aligned} \left\| \square^{-1}((|E_b| + |B_b|)\sigma_{-1}) \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}(S_T)} &\leq C_{\gamma_\varepsilon}^* \left\| (|E_b| + |B_b|)\sigma_{-1} \right\|_{L_t^{\frac{6}{5-2\varepsilon}} L_x^{\frac{3}{2+\varepsilon}}(S_T)} \\ &\leq C_{\gamma_\varepsilon}^* T^{\frac{5-2\varepsilon}{6}} C_3(\varepsilon, T) \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_T)}^2, \end{aligned}$$

as was to be shown.  $\square$

**Lemma 3.5** *For  $\varepsilon \in ]0, 1]$ ,*

$$\left\| |E_\sharp| + |B_\sharp| \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}(S_T)} \leq C_5(\varepsilon, T, \text{data}, \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_T)}),$$

where  $C_5$  is increasing in both the  $T$ -argument and the  $\|\cdot\|$ -argument.

**Proof:** The argument is similar to [18, Thm. 4.8, p. 108]. Fix an interval  $[a, b] \subset [0, T]$ . According to (3.12), (3.4) and (3.5), and (3.16),

$$\begin{aligned} &\left\| |E_\sharp| + |B_\sharp| \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}([a, b] \times \mathbb{R}^3)} \\ &\leq \left\| \square^{-1}((|E| + |B|)\sigma_{-1}) \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}([a, b] \times \mathbb{R}^3)} \\ &\leq \left\| \square^{-1}((|E_D| + |E_{DT}| + |B_D| + |B_{DT}|)\sigma_{-1}) \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}([a, b] \times \mathbb{R}^3)} \\ &\quad + \left\| \square^{-1}((|E_b| + |B_b|)\sigma_{-1}) \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}(S_T)} + \left\| \square^{-1}((|E_\sharp| + |B_\sharp|)\sigma_{-1}) \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}([a, b] \times \mathbb{R}^3)} \\ &\leq (\text{data}) + C_4(\varepsilon, T) \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_T)}^2 + \left\| \square^{-1}((|E_\sharp| + |B_\sharp|)\sigma_{-1}) \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}([a, b] \times \mathbb{R}^3)}. \end{aligned}$$

Let  $F = (|E_\sharp| + |B_\sharp|)\sigma_{-1}$  and  $u = \square^{-1}F$ . Then (3.13) for  $\gamma_\varepsilon = \frac{2}{3}(1 - \varepsilon)$  yields

$$\begin{aligned} & \|u\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}([a, b] \times \mathbb{R}^3)} \\ & \leq C_{\gamma_\varepsilon}^* \left( \|u(a)\|_{\dot{H}_x^{\gamma_\varepsilon}(\mathbb{R}^3)} + \|\partial_t u(a)\|_{\dot{H}_x^{\gamma_\varepsilon-1}(\mathbb{R}^3)} + \|F\|_{L_t^{\frac{6}{5-2\varepsilon}} L_x^{\frac{3}{2+\varepsilon}}([a, b] \times \mathbb{R}^3)} \right) \\ & \leq C_{\gamma_\varepsilon}^* \left( \|u(a)\|_{\dot{H}_x^{\gamma_\varepsilon}(\mathbb{R}^3)} + \|\partial_t u(a)\|_{\dot{H}_x^{\gamma_\varepsilon-1}(\mathbb{R}^3)} \right. \\ & \quad \left. + \left\| |E_\sharp| + |B_\sharp| \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}([a, b] \times \mathbb{R}^3)} \|\sigma_{-1}\|_{L_t^2 L_x^2([a, b] \times \mathbb{R}^3)} \right). \end{aligned}$$

As a consequence,

$$\begin{aligned} & \left\| |E_\sharp| + |B_\sharp| \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}([a, b] \times \mathbb{R}^3)} \\ & \leq (\text{data}) + C_4(\varepsilon, T) \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_T)}^2 + C_{\gamma_\varepsilon}^* \left( \|u(a)\|_{\dot{H}_x^{\gamma_\varepsilon}(\mathbb{R}^3)} + \|\partial_t u(a)\|_{\dot{H}_x^{\gamma_\varepsilon-1}(\mathbb{R}^3)} \right. \\ & \quad \left. + \left\| |E_\sharp| + |B_\sharp| \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}([a, b] \times \mathbb{R}^3)} \|\sigma_{-1}\|_{L_t^2 L_x^2([a, b] \times \mathbb{R}^3)} \right). \end{aligned} \quad (3.17)$$

Without loss of generality suppose that  $C_{\gamma_\varepsilon}^* \|\sigma_{-1}\|_{L_t^2 L_x^2(S_T)} \geq 1$ , since otherwise one can just take  $[a, b] = [0, T]$ . Fix a finite partition  $0 = T_0 < T_1 < T_2 < \dots < T_{N-1} < T_N = T$  of  $[0, T]$  such that

$$\|\sigma_{-1}\|_{L_t^2 L_x^2([T_j, T_{j+1}])} = \frac{1}{2C_{\gamma_\varepsilon}^*} \quad (j = 0, \dots, N-2), \quad \|\sigma_{-1}\|_{L_t^2 L_x^2([T_{N-1}, T_N])} \leq \frac{1}{2C_{\gamma_\varepsilon}^*}. \quad (3.18)$$

Note that

$$(N-1) \frac{1}{4(C_{\gamma_\varepsilon}^*)^2} \leq \sum_{j=0}^{N-2} \|\sigma_{-1}\|_{L_t^2 L_x^2([T_{N-1}, T_N])}^2 \leq \|\sigma_{-1}\|_{L_t^2 L_x^2(S_T)}^2$$

yields the upper bound

$$N \leq 1 + 4(C_{\gamma_\varepsilon}^*)^2 \|\sigma_{-1}\|_{L_t^2 L_x^2(S_T)}^2. \quad (3.19)$$

By (3.17) and (3.18),

$$\begin{aligned} & \left\| |E_\sharp| + |B_\sharp| \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}([T_j, T_{j+1}] \times \mathbb{R}^3)} \\ & \leq 2(\text{data})_j + 2C_4(\varepsilon, T) \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_T)}^2 + 2C_{\gamma_\varepsilon}^* \left( \|u(T_j)\|_{\dot{H}_x^{\gamma_\varepsilon}(\mathbb{R}^3)} + \|\partial_t u(T_j)\|_{\dot{H}_x^{\gamma_\varepsilon-1}(\mathbb{R}^3)} \right). \end{aligned} \quad (3.20)$$

Thus in particular

$$\begin{aligned} \|F\|_{L_t^{\frac{6}{5-2\varepsilon}} L_x^{\frac{3}{2+\varepsilon}}([T_j, T_{j+1}] \times \mathbb{R}^3)} & \leq \frac{1}{2C_{\gamma_\varepsilon}^*} \left\| |E_\sharp| + |B_\sharp| \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}([T_j, T_{j+1}] \times \mathbb{R}^3)} \\ & \leq \frac{1}{C_{\gamma_\varepsilon}^*} (\text{data})_j + \frac{1}{C_{\gamma_\varepsilon}^*} C_4(\varepsilon, T) \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_T)}^2 \\ & \quad + \|u(T_j)\|_{\dot{H}_x^{\gamma_\varepsilon}(\mathbb{R}^3)} + \|\partial_t u(T_j)\|_{\dot{H}_x^{\gamma_\varepsilon-1}(\mathbb{R}^3)}, \end{aligned}$$

so that by (3.13) for the interval  $[T_j, T_{j+1}]$ ,

$$\begin{aligned} & \|u(T_{j+1})\|_{\dot{H}_x^{\gamma_\varepsilon}(\mathbb{R}^3)} + \|\partial_t u(T_{j+1})\|_{\dot{H}_x^{\gamma_\varepsilon-1}(\mathbb{R}^3)} \\ & \leq C_{\gamma_\varepsilon}^* \left( \|u(T_j)\|_{\dot{H}_x^{\gamma_\varepsilon}(\mathbb{R}^3)} + \|\partial_t u(T_j)\|_{\dot{H}_x^{\gamma_\varepsilon-1}(\mathbb{R}^3)} + \|F\|_{L_t^{\frac{6}{5-2\varepsilon}} L_x^{\frac{3}{2+\varepsilon}}([T_j, T_{j+1}] \times \mathbb{R}^3)} \right) \\ & \leq (\text{data})_j + C_4(\varepsilon, T) \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_T)}^2 + 2C_{\gamma_\varepsilon}^* \left( \|u(T_j)\|_{\dot{H}_x^{\gamma_\varepsilon}(\mathbb{R}^3)} + \|\partial_t u(T_j)\|_{\dot{H}_x^{\gamma_\varepsilon-1}(\mathbb{R}^3)} \right). \end{aligned}$$

Iteration of this estimate and noting that  $u(0) = \partial_t u(0) = 0$  leads to the bound

$$\|u(T_j)\|_{\dot{H}_x^{\gamma_\varepsilon}(\mathbb{R}^3)} + \|\partial_t u(T_j)\|_{\dot{H}_x^{\gamma_\varepsilon-1}(\mathbb{R}^3)} \leq \sum_{i=0}^{j-1} (2C_{\gamma_\varepsilon}^*)^{j-1-i} \left( (\text{data})_i + C_4(\varepsilon, T) \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_T)}^2 \right).$$

Hence by (3.20),

$$\begin{aligned} & \left\| |E_\sharp| + |B_\sharp| \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}([T_j, T_{j+1}] \times \mathbb{R}^3)} \\ & \leq 2 \sum_{i=0}^j (2C_{\gamma_\varepsilon}^*)^{j-i} (\text{data})_i + 2C_4(\varepsilon, T) \left( \sum_{i=0}^j (2C_{\gamma_\varepsilon}^*)^i \right) \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_T)}^2 \end{aligned}$$

for  $j = 0, \dots, N-1$ . Therefore the estimate

$$\begin{aligned} & \left\| |E_\sharp| + |B_\sharp| \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}(S_T)} \\ & \leq \sum_{j=0}^{N-1} \left\| |E_\sharp| + |B_\sharp| \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}([T_j, T_{j+1}] \times \mathbb{R}^3)} \\ & \leq 2 \sum_{j=0}^{N-1} \sum_{i=0}^j (2C_{\gamma_\varepsilon}^*)^{j-i} (\text{data})_i + 2C_4(\varepsilon, T) \left( \sum_{j=0}^{N-1} \sum_{i=0}^j (2C_{\gamma_\varepsilon}^*)^i \right) \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_T)}^2 \end{aligned}$$

is obtained from (3.19). Recalling (3.19), the claim is obtained.  $\square$

**Corollary 3.6** For  $\varepsilon \in ]0, 1]$ ,

$$\left\| |E| + |B| \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}(S_T)} \leq C_6(\varepsilon, T, \text{data}, \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_T)}),$$

where  $C_6$  is increasing in both the  $T$ -argument and the  $\|\cdot\|$ -argument.

**Proof:** From (3.4) and (3.5),

$$E = E_D + E_{DT} + E_b + E_\sharp \quad \text{and} \quad B = B_D + B_{DT} + B_b + B_\sharp,$$

where  $E_D$ ,  $E_{DT}$ ,  $B_D$ , and  $B_{DT}$  are data terms. Since Corollary 3.4 implies that

$$\left\| |E_b| + |B_b| \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}(S_T)} \leq C_3(\varepsilon, T) T^{\frac{1-\varepsilon}{3}} \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_T)},$$

it remains to apply Lemma 3.5.  $\square$

**Corollary 3.7** If  $\varepsilon \in ]0, \frac{1}{10}]$ , then

$$m_{\frac{3(1-2\varepsilon)}{1+2\varepsilon}}(t) \leq C_7(0, \varepsilon, t, \text{data}, \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_t)}),$$

where  $C_7$  is increasing in both the  $t$ -argument and the  $\|\cdot\|$ -argument.

**Proof:** By Lemma 2.1 for  $k = \frac{3(1-2\varepsilon)}{1+2\varepsilon} \geq 2$ ,

$$m_{\frac{3(1-2\varepsilon)}{1+2\varepsilon}}(t) \leq m_{\frac{3(1-2\varepsilon)}{1+2\varepsilon}}(0) + C(0, \varepsilon) \int_0^t \|E(s)\|_{L_x^{\frac{6}{1+2\varepsilon}}(\mathbb{R}^3)} m_{\frac{3(1-2\varepsilon)}{1+2\varepsilon}}(s)^{\frac{5-2\varepsilon}{6}} ds.$$

By a standard differential inequality comparison theorem, this yields

$$m_{\frac{3(1-2\varepsilon)}{1+2\varepsilon}}(t)^{\frac{1+2\varepsilon}{6}} \leq m_{\frac{3(1-2\varepsilon)}{1+2\varepsilon}}(0)^{\frac{1+2\varepsilon}{6}} + C(0, \varepsilon) \left( \frac{1+2\varepsilon}{6} \right) \int_0^t \|E(s)\|_{L_x^{\frac{6}{1+2\varepsilon}}(\mathbb{R}^3)} ds,$$

so that by Corollary 3.6,

$$\begin{aligned} m_{\frac{3(1-2\varepsilon)}{1+2\varepsilon}}(t) &\leq C(0, \varepsilon, \text{data}) \left( 1 + \left[ \int_0^t \|E(s)\|_{L_x^{\frac{6}{1+2\varepsilon}}(\mathbb{R}^3)} ds \right]^{\frac{6}{1+2\varepsilon}} \right) \\ &\leq C(0, \varepsilon, \text{data}) \left( 1 + t^{\frac{2(2+\varepsilon)}{1+2\varepsilon}} \|E\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}(S_t)}^{\frac{6}{1+2\varepsilon}} \right) \\ &\leq C(0, \varepsilon, \text{data}) \left( 1 + t^{\frac{2(2+\varepsilon)}{1+2\varepsilon}} C_6(\varepsilon, t, \text{data}, \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_t)})^{\frac{6}{1+2\varepsilon}} \right). \end{aligned}$$

Hence  $C_7$  can be defined appropriately. □

**Corollary 3.8** *If  $\varepsilon \in ]0, \frac{1}{10}]$ , then*

$$\|\sigma_{-1}(t)\|_{L_x^{\frac{4(1-\varepsilon)}{1+2\varepsilon}}(\mathbb{R}^3)} \leq C_8(0, \varepsilon, t, \text{data}, \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_t)}) P_\infty(t),$$

where  $C_8$  is increasing in both the  $t$ -argument and the  $\|\cdot\|$ -argument.

**Proof:** In Lemma 2.2 take  $q = \frac{4(1-\varepsilon)}{1+2\varepsilon}$  and  $\alpha = \frac{3(1-2\varepsilon)}{8(1-\varepsilon)}$ . Invoking Corollary 3.7, it follows that

$$\begin{aligned} \|\sigma_{-1}(t)\|_{L_x^{\frac{4(1-\varepsilon)}{1+2\varepsilon}}(\mathbb{R}^3)} &\leq C(0, \varepsilon) P_\infty(t) m_{\frac{3(1-2\varepsilon)}{1+2\varepsilon}}(t)^{\frac{1+2\varepsilon}{4(1-\varepsilon)}} \\ &\leq C(0, \varepsilon) P_\infty(t) C_7(0, \varepsilon, t, \text{data}, \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_t)})^{\frac{1+2\varepsilon}{4(1-\varepsilon)}}. \end{aligned}$$

Thus it remains to choose  $C_8$  in a suitable manner. □

## 4 Proof of Theorem 1.4

Fix any characteristic  $(X_0, P_0)$  in the support of  $f$ , i.e.,

$$f(s, X_0(s), P_0(s)) = f^{(0)}(X_0(0), P_0(0)) \neq 0, \quad s \in [0, T_{\max}[. \quad (4.1)$$

The relation

$$\begin{aligned} \frac{d}{ds} \sqrt{1 + P_0(s)^2} &= V_0(s) \cdot \dot{P}_0(s) = V_0(s) \cdot \left( E(s, X_0(s)) + V_0(s) \wedge B(s, X_0(s)) \right) \\ &= V_0(s) \cdot E(s, X_0(s)) \end{aligned}$$

in conjunction with (3.4) yields

$$\begin{aligned}
|P_0(t)| &\leq \sqrt{1 + P_0(t)^2} = \sqrt{1 + P_0(0)^2} + \int_0^t V_0(s) \cdot E(s, X_0(s)) ds \\
&= \sqrt{1 + P_0(0)^2} + \int_0^t V_0(s) \cdot (E_D + E_{DT})(s, X_0(s)) ds \\
&\quad + \int_0^t V_0(s) \cdot E_b(s, X_0(s)) ds + \int_0^t V_0(s) \cdot E_\#(s, X_0(s)) ds.
\end{aligned} \tag{4.2}$$

By the definition of  $E_b$ ,

$$\begin{aligned}
I_b(t) &= \int_0^t V_0(s) \cdot E_b(s, X_0(s)) ds \\
&= - \int_0^t ds \int_{|y| \leq s} \frac{dy}{|y|^2} \int_{\mathbb{R}^3} \frac{dp}{1 + p^2} \frac{1}{(1 + v \cdot \omega)^2} V_0(s) \cdot (v + \omega) f(s - |y|, X_0(s) + y, p) \\
&= - \int_0^t d\tau \int_\tau^t ds \int_{|\omega|=1} dS(\omega) \\
&\quad \times \int_{\mathbb{R}^3} \frac{dp}{1 + p^2} \frac{1}{(1 + v \cdot \omega)^2} V_0(s) \cdot (v + \omega) f(\tau, X_0(s) + (s - \tau)\omega, p).
\end{aligned}$$

Next write  $V_0(s) \cdot (v + \omega) = (V_0(s) + \omega) \cdot (v + \omega) - (1 + v \cdot \omega)$  and split the integral accordingly as  $I_b(t) = I_{b,1}(t) + I_{b,2}(t)$ . Firstly,

$$\begin{aligned}
|I_{b,2}(t)| &= \int_0^t d\tau \int_\tau^t ds \int_{|\omega|=1} dS(\omega) \int_{\mathbb{R}^3} \frac{dp}{1 + p^2} \frac{1}{1 + v \cdot \omega} f(\tau, X_0(s) + (s - \tau)\omega, p) \\
&\leq \mathcal{L}(0) \int_0^t d\tau \int_\tau^t ds \int_{|\omega|=1} dS(\omega) \int_{|p| \leq P_\infty(\tau)} \frac{dp}{1 + p^2} \frac{1}{1 + v \cdot \omega} \\
&\leq C(0) \int_0^t (t - \tau) \ln P_\infty(\tau) P_\infty(\tau) d\tau \leq C(0) t \int_0^t \ln P_\infty(\tau) P_\infty(\tau) d\tau,
\end{aligned}$$

where we used Lemma 2.3(a). Concerning  $I_{b,1}(t)$ ,

$$\begin{aligned}
|I_{b,1}(t)| &= \left| \int_0^t d\tau \int_\tau^t ds \int_{|\omega|=1} dS(\omega) \right. \\
&\quad \times \left. \int_{\mathbb{R}^3} \frac{dp}{1 + p^2} \frac{1}{(1 + v \cdot \omega)^2} (V_0(s) + \omega) \cdot (v + \omega) f(\tau, X_0(s) + (s - \tau)\omega, p) \right| \\
&\leq \int_0^t d\tau \int_\tau^t ds \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta \\
&\quad \times \int_{\mathbb{R}^3} \frac{dp}{1 + p^2} \frac{(1 + V_0(s) \cdot \omega)^{1/2}}{(1 + v \cdot \omega)^{3/2}} f(\tau, X_0(s) + (s - \tau)\omega, p)
\end{aligned}$$

for  $\omega = (\cos \varphi \sin \theta, \sin \varphi \sin \theta, \cos \theta)$ ; recall (3.8). If  $tP_\infty(t) \geq 1$ , then the  $\int_0^t d\tau \int_\tau^t ds$  is split to find

$$\mathbf{1}_{\{tP_\infty(t) \geq 1\}} |I_{b,1}(t)| \leq I_{b,11}(t) + I_{b,12}(t) + I_{b,13}(t),$$



where

$$\begin{aligned}
I_{b,11}(t) &= \int_0^{t-P_\infty(t)^{-1}} d\tau \int_\tau^{\tau+P_\infty(t)^{-1}} ds (\dots), \\
I_{b,12}(t) &= \int_0^{t-P_\infty(t)^{-1}} d\tau \int_{\tau+P_\infty(t)^{-1}}^t ds (\dots), \\
I_{b,13}(t) &= \int_{t-P_\infty(t)^{-1}}^t d\tau \int_\tau^t ds (\dots).
\end{aligned}$$

To begin with, by Lemma 2.3(b) for  $\theta = 1$  and  $\kappa = \frac{3}{2}$ ,

$$\begin{aligned}
I_{b,11}(t) &= \int_0^{t-P_\infty(t)^{-1}} d\tau \int_\tau^{\tau+P_\infty(t)^{-1}} ds \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta \\
&\quad \times \int_{\mathbb{R}^3} \frac{dp}{1+p^2} \frac{(1+V_0(s) \cdot \omega)^{1/2}}{(1+v \cdot \omega)^{3/2}} f(\tau, X_0(s) + (s-\tau)\omega, p) \\
&\leq C\mathcal{L}(0) P_\infty(t)^{-1} \int_0^{t-P_\infty(t)^{-1}} d\tau \max_{|\omega|=1} \int_{|p| \leq P_\infty(\tau)} \frac{dp}{1+p^2} \frac{1}{(1+v \cdot \omega)^{3/2}} \\
&\leq C(0) P_\infty(t)^{-1} \int_0^t P_\infty(\tau)^2 d\tau \leq C(0) \int_0^t P_\infty(\tau) d\tau.
\end{aligned}$$

Note that in the last step it was used that  $P_\infty(\tau) \leq P_\infty(t)$ , since  $P_\infty$  is increasing by definition. Similarly,

$$\begin{aligned}
I_{b,13}(t) &= \int_{t-P_\infty(t)^{-1}}^t d\tau \int_\tau^t ds \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta \\
&\quad \times \int_{\mathbb{R}^3} \frac{dp}{1+p^2} \frac{(1+V_0(s) \cdot \omega)^{1/2}}{(1+v \cdot \omega)^{3/2}} f(\tau, X_0(s) + (s-\tau)\omega, p) \\
&\leq \mathcal{L}(0) \int_{t-P_\infty(t)^{-1}}^t d\tau (t-\tau) \max_{|\omega|=1} \int_{|p| \leq P_\infty(\tau)} \frac{dp}{1+p^2} \frac{1}{(1+v \cdot \omega)^{3/2}} \\
&\leq C\mathcal{L}(0) \int_{t-P_\infty(t)^{-1}}^t d\tau (t-\tau) P_\infty(\tau)^2 \\
&\leq C(0) P_\infty(t)^{-1} \int_0^t P_\infty(\tau)^2 d\tau \leq C(0) \int_0^t P_\infty(\tau) d\tau.
\end{aligned}$$

It remains to deal with

$$\begin{aligned}
I_{b,12}(t) &= \int_0^{t-P_\infty(t)^{-1}} d\tau \int_{\tau+P_\infty(t)^{-1}}^t ds \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta \\
&\quad \times \int_{\mathbb{R}^3} \frac{dp}{1+p^2} \frac{(1+V_0(s) \cdot \omega)^{1/2}}{(1+v \cdot \omega)^{3/2}} f(\tau, X_0(s) + (s-\tau)\omega, p).
\end{aligned}$$

Let  $M_\tau = [\tau + P_\infty(t)^{-1}, t] \times [0, 2\pi] \times [0, \pi]$  and consider the mapping

$$\Phi_\tau : M_\tau \ni (s, \varphi, \theta) \mapsto y = X_0(s) + (s-\tau)\omega \in \mathbb{R}^3. \quad (4.3)$$

According to [15, Lemma 2.1],  $\Phi_\tau$  is a diffeomorphism and such that

$$dy = (1 + V_0(s) \cdot \omega)(s - \tau)^2 \sin \theta ds d\varphi d\theta.$$

Writing the inverse mapping as  $s = s(y)$  and  $\omega = \omega(y)$ , this yields using Lemma 2.3(b) for  $\theta = \kappa = 3$ ,

$$\begin{aligned} I_{b,12}(t) &= \int_0^{t-P_\infty(t)^{-1}} d\tau \int_{\Phi_\tau(M_\tau)} dy \frac{1}{(s-\tau)^2} \frac{1}{(1+V_0(s) \cdot \omega)^{1/2}} \\ &\quad \times \int_{|p| \leq P_\infty(\tau)} \frac{dp}{1+p^2} \frac{1}{(1+v \cdot \omega)^{3/2}} f(\tau, y, p) \\ &\leq \int_0^{t-P_\infty(t)^{-1}} d\tau \left( \int_{\Phi_\tau(M_\tau)} dy \frac{1}{(s-\tau)^4} \frac{1}{(1+V_0(s) \cdot \omega)} \int_{|p| \leq P_\infty(\tau)} \frac{dp}{(1+p^2)^3} \frac{1}{(1+v \cdot \omega)^3} \right)^{1/2} \\ &\quad \times \left( \int_{\Phi_\tau(M_\tau)} \int_{|p| \leq P_\infty(\tau)} dy dp (1+p^2) f(\tau, y, p)^2 \right)^{1/2} \\ &\leq C\mathcal{L}(0)^{1/2} \int_0^{t-P_\infty(t)^{-1}} d\tau \left( \int_{\Phi_\tau(M_\tau)} dy \frac{1}{(s-\tau)^4} \frac{1}{(1+V_0(s) \cdot \omega)} \right)^{1/2} P_\infty(\tau)^{1/2} m_2(\tau)^{1/2} \\ &\leq C(0) \int_0^{t-P_\infty(t)^{-1}} d\tau \left( \int_{\tau+P_\infty(t)^{-1}}^t ds \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta \frac{1}{(s-\tau)^2} \right)^{1/2} P_\infty(\tau)^{1/2} m_2(\tau)^{1/2} \\ &\leq C(0) P_\infty(t)^{1/2} \int_0^t P_\infty(\tau)^{1/2} m_2(\tau)^{1/2} d\tau. \end{aligned}$$

To summarize, it has been shown that

$$\begin{aligned} \mathbf{1}_{\{tP_\infty(t) \geq 1\}} |I_b(t)| &\leq \mathbf{1}_{\{tP_\infty(t) \geq 1\}} |I_{b,11}(t)| + \mathbf{1}_{\{tP_\infty(t) \geq 1\}} |I_{b,12}(t)| + \mathbf{1}_{\{tP_\infty(t) \geq 1\}} |I_{b,13}(t)| \\ &\quad + \mathbf{1}_{\{tP_\infty(t) \geq 1\}} |I_{b,2}(t)| \\ &\leq C(0) \int_0^t P_\infty(\tau) d\tau + C(0) P_\infty(t)^{1/2} \int_0^t P_\infty(\tau)^{1/2} m_2(\tau)^{1/2} d\tau \\ &\quad + C(0) t \int_0^t \ln P_\infty(\tau) P_\infty(\tau) d\tau \\ &\leq C(0) P_\infty(t)^{1/2} \int_0^t P_\infty(\tau)^{1/2} m_2(\tau)^{1/2} d\tau \\ &\quad + C(0) (1+t) \int_0^t \ln P_\infty(\tau) P_\infty(\tau) d\tau. \end{aligned} \tag{4.4}$$

Next, by the definition of  $E_\sharp$ ,

$$\begin{aligned} I_\sharp(t) &= \int_0^t V_0(s) \cdot E_\sharp(s, X_0(s)) ds \\ &= - \int_0^t ds \int_{|y| \leq s} \frac{dy}{|y|} \int_{\mathbb{R}^3} dp V_0(s) \cdot K_{E, \sharp}(\omega, v) (Lf)(s - |y|, X_0(s) + y, p). \end{aligned}$$

From (3.7) in Lemma 3.1 it hence follows that

$$|I_\sharp(t)| \leq \int_0^t ds \int_{|y| \leq s} \frac{dy}{|y|} \int_{\mathbb{R}^3} \frac{dp}{\sqrt{1+p^2}} \frac{1}{1+v \cdot \omega} (|E| + |B|)(s - |y|, X_0(s) + y)$$

$$\begin{aligned}
& \times f(s - |y|, X_0(s) + y, p) \\
& \leq \int_0^t ds \int_{|y| \leq s} \frac{dy}{|y|} (|E| + |B|)(s - |y|, X_0(s) + y) \sigma_{-1}(s - |y|, X_0(s) + y) \\
& = \int_0^t d\tau \int_\tau^t ds (s - \tau) \int_{|\omega|=1} dS(\omega) (|E| + |B|)(\tau, X_0(s) + (s - \tau)\omega) \\
& \quad \times \sigma_{-1}(\tau, X_0(s) + (s - \tau)\omega).
\end{aligned}$$

Next the transformation  $\Phi_\tau$  from (4.3) is used on  $M_\tau = [\tau, t] \times [0, 2\pi] \times [0, \pi]$ . This yields

$$\begin{aligned}
|I_\sharp(t)| & \leq \int_0^t d\tau \int_\tau^t ds (s - \tau) \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta (|E| + |B|)(\tau, X_0(s) + (s - \tau)\omega) \\
& \quad \times \sigma_{-1}(\tau, X_0(s) + (s - \tau)\omega) \\
& = \int_0^t d\tau \int_{\Phi_\tau(M_\tau)} dy \frac{1}{(s - \tau)} \frac{1}{(1 + V_0(s) \cdot \omega)} (|E| + |B|)(\tau, y) \sigma_{-1}(\tau, y)
\end{aligned}$$

for  $s = s(y)$  and  $\omega = \omega(y)$ . Now fix  $\varepsilon \in ]0, \frac{1}{20}]$  and define  $\alpha_\varepsilon = \frac{12(1-\varepsilon)}{7-20\varepsilon+4\varepsilon^2} \in ]1, 2[$ . The general Hölder inequality in  $y$  for the exponents  $(\alpha_\varepsilon, \frac{6}{1+2\varepsilon}, \frac{4(1-\varepsilon)}{1+2\varepsilon})$  in conjunction with (4.3) and Lemma 2.3(c) implies that

$$\begin{aligned}
|I_\sharp(t)| & \leq \int_0^t d\tau \left( \int_{\Phi_\tau(M_\tau)} dy \frac{1}{(s - \tau)^{\alpha_\varepsilon}} \frac{1}{(1 + V_0(s) \cdot \omega)^{\alpha_\varepsilon}} \right)^{1/\alpha_\varepsilon} \\
& \quad \times \|(|E| + |B|)(\tau)\|_{L_x^{\frac{6}{1+2\varepsilon}}(\mathbb{R}^3)} \|\sigma_{-1}(\tau)\|_{L_x^{\frac{4(1-\varepsilon)}{1+2\varepsilon}}(\mathbb{R}^3)} \\
& = \int_0^t d\tau \left( \int_\tau^t ds \int_0^{2\pi} d\varphi \int_0^\pi d\theta \sin \theta \frac{(s - \tau)^{2-\alpha_\varepsilon}}{(1 + V_0(s) \cdot \omega)^{\alpha_\varepsilon-1}} \right)^{1/\alpha_\varepsilon} \\
& \quad \times \|(|E| + |B|)(\tau)\|_{L_x^{\frac{6}{1+2\varepsilon}}(\mathbb{R}^3)} \|\sigma_{-1}(\tau)\|_{L_x^{\frac{4(1-\varepsilon)}{1+2\varepsilon}}(\mathbb{R}^3)} \\
& \leq t^{\frac{2}{\alpha_\varepsilon}-1} \int_0^t d\tau \left( \int_\tau^t ds \int_{|\omega|=1} \frac{dS(\omega)}{(1 + V_0(s) \cdot \omega)^{\alpha_\varepsilon-1}} \right)^{1/\alpha_\varepsilon} \\
& \quad \times \|(|E| + |B|)(\tau)\|_{L_x^{\frac{6}{1+2\varepsilon}}(\mathbb{R}^3)} \|\sigma_{-1}(\tau)\|_{L_x^{\frac{4(1-\varepsilon)}{1+2\varepsilon}}(\mathbb{R}^3)} \\
& \leq C(\varepsilon) t^{\frac{3}{\alpha_\varepsilon}-1} \int_0^t \|(|E| + |B|)(\tau)\|_{L_x^{\frac{6}{1+2\varepsilon}}(\mathbb{R}^3)} \|\sigma_{-1}(\tau)\|_{L_x^{\frac{4(1-\varepsilon)}{1+2\varepsilon}}(\mathbb{R}^3)} d\tau.
\end{aligned}$$

Returning to (4.2), it follows from this estimate and (4.4) that

$$\begin{aligned}
\mathbf{1}_{\{tP_\infty(t) \geq 1\}} |P_0(t)| & \leq \sqrt{1 + P_0(0)^2} + \int_0^t V_0(s) \cdot (E_D + E_{DT})(s, X_0(s)) ds \\
& \quad + \mathbf{1}_{\{tP_\infty(t) \geq 1\}} |I_b(t)| + \mathbf{1}_{\{tP_\infty(t) \geq 1\}} |I_\sharp(t)| \\
& \leq \sqrt{1 + P_0(0)^2} + (\text{data}) + C(0) P_\infty(t)^{1/2} \int_0^t P_\infty(\tau)^{1/2} m_2(\tau)^{1/2} d\tau \\
& \quad + C(0) (1 + t) \int_0^t \ln P_\infty(\tau) P_\infty(\tau) d\tau \\
& \quad + C(\varepsilon) t^{\frac{3}{\alpha_\varepsilon}-1} \int_0^t \|(|E| + |B|)(\tau)\|_{L_x^{\frac{6}{1+2\varepsilon}}(\mathbb{R}^3)} \|\sigma_{-1}(\tau)\|_{L_x^{\frac{4(1-\varepsilon)}{1+2\varepsilon}}(\mathbb{R}^3)} d\tau.
\end{aligned}$$

Since  $(X_0, P_0)$  is in the support of  $f$ ,  $\sqrt{1 + P_0(0)^2} \leq (\text{data})$  uniformly in the characteristic, as  $f^{(0)} \in C_0^1(\mathbb{R}^3 \times \mathbb{R}^3)$  is compactly supported in  $p$  by assumption; also see (4.1). If  $f(s, x, p) \neq 0$ , then  $f(s, x, p) = f(s, X_0(s), P_0(s))$  for a characteristic  $(X_0, P_0)$  in the support of  $f$ . It follows from the preceding estimate that

$$\begin{aligned} \mathbf{1}_{\{tP_\infty(t) \geq 1\}} P_\infty(t) &\leq (\text{data}) + C(0)P_\infty(t)^{1/2} \int_0^t P_\infty(\tau)^{1/2} m_2(\tau)^{1/2} d\tau \\ &\quad + C(0)(1+t) \int_0^t \ln P_\infty(\tau) P_\infty(\tau) d\tau \\ &\quad + C(\varepsilon) t^{\frac{3}{\alpha_\varepsilon}-1} \int_0^t \|(|E| + |B|)(\tau)\|_{L_x^{\frac{6}{1+2\varepsilon}}(\mathbb{R}^3)} \|\sigma_{-1}(\tau)\|_{L_x^{\frac{4(1-\varepsilon)}{1+2\varepsilon}}(\mathbb{R}^3)} d\tau \end{aligned} \quad (4.5)$$

for  $t \in [0, T_{\max}]$ . Now suppose that  $T_{\max} < \infty$ . Then by the assumption (1.11),

$$\|\sigma_{-1}\|_{L_t^\infty L_x^2(S_T)} \leq \varpi(T) \leq \max_{T' \in [0, T_{\max}]} \varpi(T') =: \varpi_{\max} < \infty$$

for all  $T \in [0, T_{\max}]$ . Thus according to Corollaries 3.6, 3.8, and 3.7 for  $T, t \in [0, T_{\max}]$ ,

$$\begin{aligned} \left\| |E| + |B| \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}(S_T)} &\leq C_6(\varepsilon, T, \text{data}, \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_T)}) \\ &\leq C_6(\varepsilon, T_{\max}, \text{data}, \varpi_{\max}), \\ \|\sigma_{-1}(t)\|_{L_x^{\frac{4(1-\varepsilon)}{1+2\varepsilon}}(\mathbb{R}^3)} &\leq C_8(0, \varepsilon, t, \text{data}, \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_t)}) P_\infty(t) \\ &\leq C_8(0, \varepsilon, T_{\max}, \text{data}, \varpi_{\max}) P_\infty(t), \\ m_2(t) &\leq m_{\frac{3(1-2\varepsilon)}{1+2\varepsilon}}(t) \leq C_7(0, \varepsilon, t, \text{data}, \|\sigma_{-1}\|_{L_t^\infty L_x^2(S_t)}) \\ &\leq C_7(0, \varepsilon, T_{\max}, \text{data}, \varpi_{\max}). \end{aligned} \quad (4.6)$$

Henceforth the dependence of the constants on  $\mathcal{L}(0)$ , the fixed  $\varepsilon$ , and the initial data is suppressed, and only the dependence on  $T_{\max}$  and  $\varpi_{\max}$  is made explicit. Thus (4.5) leads to

$$\begin{aligned} \mathbf{1}_{\{tP_\infty(t) \geq 1\}} P_\infty(t) &\leq (\text{data}) + C(T_{\max}, \varpi_{\max}) P_\infty(t)^{1/2} \int_0^t P_\infty(\tau)^{1/2} d\tau \\ &\quad + C(0)(1+T_{\max}) \int_0^t \ln P_\infty(\tau) P_\infty(\tau) d\tau \\ &\quad + C(T_{\max}, \varpi_{\max}) T_{\max}^{\frac{3}{\alpha_\varepsilon}-1} \int_0^t \|(|E| + |B|)(\tau)\|_{L_x^{\frac{6}{1+2\varepsilon}}(\mathbb{R}^3)} P_\infty(\tau) d\tau \end{aligned}$$

for  $t \in [0, T_{\max}]$ . Since  $P_\infty(\tau) \leq P_\infty(t)^{1/2} P_\infty(\tau)^{1/2}$ , it follows that for a certain constant  $C_2 = C_2(T_{\max}, \varpi_{\max}) > 0$ ,

$$\begin{aligned} &\mathbf{1}_{\{tP_\infty(t) \geq 1\}} P_\infty(t)^{1/2} \\ &\leq (\text{data}) + C(T_{\max}, \varpi_{\max}) \int_0^t P_\infty(\tau)^{1/2} d\tau \\ &\quad + C(T_{\max}, \varpi_{\max}) \int_0^t \ln P_\infty(\tau)^{1/2} P_\infty(\tau)^{1/2} d\tau \\ &\quad + C(T_{\max}, \varpi_{\max}) \int_0^t \|(|E| + |B|)(\tau)\|_{L_x^{\frac{6}{1+2\varepsilon}}(\mathbb{R}^3)} P_\infty(\tau)^{1/2} d\tau \\ &\leq (\text{data}) + C_2(T_{\max}, \varpi_{\max}) \int_0^t \left(1 + \|(|E| + |B|)(\tau)\|_{L_x^{\frac{6}{1+2\varepsilon}}(\mathbb{R}^3)}\right) \ln P_\infty(\tau)^{1/2} P_\infty(\tau)^{1/2} d\tau \end{aligned}$$

for  $t \in [0, T_{\max}[$ . By the local existence theorem (Theorem 1.1), there is a constant  $C_1 > 0$  such that  $\max_{t \in [0, T_{\max}/2]} P_{\infty}(t) = P_{\infty}(T_{\max}/2) \leq C_1$ . Hence if  $tP_{\infty}(t) \leq 1$  and  $t \in [0, T_{\max}/2]$ , then

$$P_{\infty}(t)^{1/2} \leq C_1^{1/2}.$$

On the other hand, if  $tP_{\infty}(t) \leq 1$  and  $t \in [T_{\max}/2, T_{\max}[$ , then

$$P_{\infty}(t)^{1/2} \leq \frac{1}{\sqrt{t}} \leq \left( \frac{2}{T_{\max}} \right)^{1/2}.$$

Therefore

$$\begin{aligned} P_{\infty}(t)^{1/2} &\leq C_1(T_{\max}, \varpi_{\max}) \\ &\quad + C_2(T_{\max}, \varpi_{\max}) \int_0^t \left( 1 + \|(|E| + |B|)(\tau)\|_{L_x^{\frac{6}{1+2\varepsilon}}(\mathbb{R}^3)} \right) \ln P_{\infty}(\tau)^{1/2} P_{\infty}(\tau)^{1/2} d\tau \end{aligned}$$

for  $t \in [0, T_{\max}[$ , where  $C_1(T_{\max}, \varpi_{\max}) = (\text{data}) + C_1^{1/2} + (\frac{2}{T_{\max}})^{1/2}$ . This integral inequality and (4.6) imply that

$$\begin{aligned} \ln P_{\infty}(t)^{1/2} &\leq C(T_{\max}, \varpi_{\max}) \exp \left( \int_0^t \left( 1 + \|(|E| + |B|)(\tau)\|_{L_x^{\frac{6}{1+2\varepsilon}}(\mathbb{R}^3)} \right) d\tau \right) \\ &\leq C(T_{\max}, \varpi_{\max}) \exp \left( t + t^{\frac{2+\varepsilon}{3}} \left\| |E| + |B| \right\|_{L_t^{\frac{3}{1-\varepsilon}} L_x^{\frac{6}{1+2\varepsilon}}(S_t)} \right) \\ &\leq C(T_{\max}, \varpi_{\max}) \exp \left( T_{\max} + T_{\max}^{\frac{2+\varepsilon}{3}} C_6(\varepsilon, T_{\max}, \text{data}, \varpi_{\max}) \right) \\ &\leq C_3(T_{\max}, \varpi_{\max}) \end{aligned}$$

for  $t \in [0, T_{\max}[$ . Defining  $\varpi_1(t) = \exp(C_3(T_{\max}, \varpi_{\max}))^2$ , the criterion (1.8) in Theorem 1.1 is verified for  $\varpi_1$ . From this result it hence follows that  $T_{\max} = \infty$ , which is a contradiction to what was supposed before. As a consequence,  $T_{\max} = \infty$  must be satisfied and the proof of Theorem 1.4 is complete.  $\square$

## 5 Proof of Corollary 1.5

**Lemma 5.1** Define  $\sigma_{-1}$  by (1.10) and  $\mathcal{I}_{\theta}$  by (1.9). Then for every  $a \in [0, \infty[$  and  $\varepsilon > 0$  there is a constant  $C = C(0, a, \varepsilon) > 0$  such that

$$\sigma_{-1}(t, x) \leq C \left( 1 + \mathcal{I}_{a+1}(t, x)^{\frac{2+\varepsilon a}{2+a}} \right). \quad (5.1)$$

**Proof:** Fix  $\omega \in \mathbb{R}^3$  such that  $|\omega| = 1$ . Since  $1 + v \cdot \omega \geq 1 - |v| \geq \frac{1}{2(1+p^2)}$ , it follows for  $R \in [10, \infty[$  and  $\varepsilon \in ]0, 2]$  that

$$\begin{aligned} &\int_{\mathbb{R}^3} \frac{dp}{\sqrt{1+p^2}} \frac{1}{(1+v \cdot \omega)} f \\ &= \int_{|p| \leq R, 1+v \cdot \omega \leq \varepsilon} \frac{dp}{\sqrt{1+p^2}} \frac{1}{(1+v \cdot \omega)} f + \int_{|p| \leq R, 1+v \cdot \omega > \varepsilon} \frac{dp}{\sqrt{1+p^2}} \frac{1}{(1+v \cdot \omega)} f \end{aligned}$$

$$\begin{aligned}
& + \int_{|p|>R} \frac{dp}{\sqrt{1+p^2}} \frac{1}{(1+v \cdot \omega)} f \\
\leq & 2\mathcal{L}(0) \int_{|p|\leq R, 1+v \cdot \omega \leq \varepsilon} dp \sqrt{1+p^2} + \mathcal{L}(0) \int_{|p|\leq R, 1+v \cdot \omega > \varepsilon} \frac{dp}{\sqrt{1+p^2}} \frac{1}{(1+v \cdot \omega)} \\
& + 2 \int_{|p|>R} dp \sqrt{1+p^2} f
\end{aligned}$$

For the first integral, the transformation (2.3) yields

$$\begin{aligned}
\int_{|p|\leq R, 1+v \cdot \omega \leq \varepsilon} dp \sqrt{1+p^2} &= \int_{|p|\leq R, 1+v_3 \leq \varepsilon} dp \sqrt{1+p^2} \\
&\leq C \int_0^R dr r^2 \sqrt{1+r^2} \int_0^\pi d\theta \sin \theta \mathbf{1}_{\{1+\frac{r \cos \theta}{\sqrt{1+r^2}} \leq \varepsilon\}} \\
&\leq C \int_0^{R^b} \frac{d\sigma \sigma^2}{(1-\sigma^2)^3} \int_{-1}^1 ds \mathbf{1}_{\{1+\sigma s \leq \varepsilon\}} \\
&\leq C\varepsilon \int_0^{R^b} \frac{d\sigma \sigma}{(1-\sigma^2)^3} \leq C\varepsilon R^4.
\end{aligned}$$

Similarly, the second integral can be bounded by

$$\begin{aligned}
\int_{|p|\leq R, 1+v \cdot \omega > \varepsilon} \frac{dp}{\sqrt{1+p^2}} \frac{1}{(1+v \cdot \omega)} &\leq C \int_0^{R^b} \frac{d\sigma \sigma^2}{(1-\sigma^2)^2} \int_{-1}^1 \frac{ds}{1+\sigma s} \mathbf{1}_{\{1+\sigma s > \varepsilon\}} \\
&\leq C \int_0^{R^b} \frac{d\sigma \sigma}{(1-\sigma^2)^2} \int_\varepsilon^2 \frac{d\tau}{\tau} \leq C \ln \left( \frac{2}{\varepsilon} \right) R^2.
\end{aligned}$$

Hence for  $a \in [0, \infty[$ ,

$$\sigma_{-1} \leq C(0)\varepsilon R^4 + C(0) \ln \left( \frac{2}{\varepsilon} \right) R^2 + 2R^{-a} \int_{|p|>R} (1+p^2)^{\frac{a+1}{2}} f dp.$$

Upon choosing  $\varepsilon = 1/R^2$ , this leads to

$$\sigma_{-1} \leq C(0) \ln(R) R^2 + 2R^{-a} \int_{\mathbb{R}^3} (1+p^2)^{\frac{a+1}{2}} f dp. \quad (5.2)$$

Furthermore,  $\sigma_{-1} \leq 2 \int_{\mathbb{R}^3} \sqrt{1+p^2} f dp$  is always satisfied. Next fix a constant  $C_* = C_*(a)$  such that

$$I^{\frac{1}{2+a}} (\ln(10+I))^{-\frac{1}{2+a}} \geq 10, \quad I \geq C_*(a).$$

If  $\mathcal{I}_{a+1} \leq C_*(a)$ , then

$$\sigma_{-1} \leq 2 \int_{\mathbb{R}^3} \sqrt{1+p^2} f dp \leq 2\mathcal{I}_{a+1} \leq 2C_*(a).$$

On the other hand, if  $\mathcal{I}_{a+1} \geq C_*(a)$ , then take  $R = \mathcal{I}_{a+1}^{\frac{1}{2+a}} (\ln(10+\mathcal{I}_{a+1}))^{-\frac{1}{2+a}} \geq 10$  in (5.2) to obtain

$$\begin{aligned}
\sigma_{-1} &\leq C(0) \ln(R) R^2 + 2R^{-a} \mathcal{I}_{a+1} \\
&\leq C(0, a) \mathcal{I}_{a+1}^{\frac{2}{2+a}} \left( [\ln \mathcal{I}_{a+1} - \ln \ln(10 + \mathcal{I}_{a+1})] (\ln(10 + \mathcal{I}_{a+1}))^{-\frac{2}{2+a}} + (\ln(10 + \mathcal{I}_{a+1}))^{\frac{a}{2+a}} \right) \\
&\leq C(0, a) \mathcal{I}_{a+1}^{\frac{2}{2+a}} (\ln(10 + \mathcal{I}_{a+1}))^{\frac{a}{2+a}}.
\end{aligned}$$

If  $\varepsilon > 0$  is fixed, then select  $C = C(a, \varepsilon)$  such that  $\ln(10 + I) \leq C(a, \varepsilon)I^\varepsilon$  whenever  $I \geq C_*(a)$ . Therefore  $\mathcal{I}_{a+1} \geq C_*(a)$  yields

$$\sigma_{-1} \leq C(0, a) C(a, \varepsilon)^{\frac{a}{2+a}} \mathcal{I}_{a+1}^{\frac{2+\varepsilon a}{2+a}},$$

and hence (5.1). □

**Corollary 5.2** *For every  $a \in [0, \infty[$  and  $\varepsilon > 0$  there is a constant  $C = C(0, a, \varepsilon) > 0$  such that*

$$\|\sigma_{-1}(t)\|_{L_x^2(\mathbb{R}^3)}^2 \leq C \left( 1 + t^3 + \|\mathcal{I}_{a+1}(t)\|_{L_x^q(\mathbb{R}^3)}^q \right)$$

for  $q = \frac{2(2+\varepsilon a)}{2+a}$ .

**Proof:** Let  $R_0$  be fixed such that  $f^{(0)}(x, p) = 0$  for  $|x| \geq R_0$ . Then (1.1) and (2.1) implies that  $f(t, x, p) = 0$  for  $|x| \geq R_0 + t$ . In particular,  $\sigma_{-1}(t, x) = 0$  for  $|x| \geq R_0 + t$ . Hence squaring and integrating (5.1) it follows that

$$\int_{\mathbb{R}^3} \sigma_{-1}(t, x)^2 dx \leq C \left( (R_0 + t)^3 + \int_{\mathbb{R}^3} \mathcal{I}_{a+1}(t, x)^{\frac{2(2+\varepsilon a)}{2+a}} dx \right),$$

which yields the claim. □

**Proof of Corollary 1.5 :** Let  $\theta > 1$  and  $q \in ]\frac{4}{\theta+1}, \infty[$  be given, and suppose that there is a function  $\varpi_7 \in C([0, \infty[)$  such that  $\|\mathcal{I}_\theta(t)\|_{L_x^q(\mathbb{R}^3)} \leq \varpi_7(t)$  is verified for  $t \in [0, T_{\max}[$ . Defining

$$a = \theta - 1, \quad \varepsilon = \frac{q(2+a)}{2a} - \frac{2}{a},$$

we have  $a > 0$  and  $\varepsilon > 0$ , and in addition  $q = \frac{2(2+\varepsilon a)}{2+a}$ . Thus we can apply Corollary 5.2 to deduce that for  $t \in [0, T_{\max}[$ :

$$\|\sigma_{-1}(t)\|_{L_x^2(\mathbb{R}^3)}^2 \leq C \left( 1 + t^3 + \|\mathcal{I}_\theta(t)\|_{L_x^q(\mathbb{R}^3)}^q \right) \leq C \left( 1 + t^3 + \varpi_7(t)^q \right).$$

Hence Theorem 1.4 applies. □

## References

- [1] BOUCHUT F., GOLSE F. & PALLARD C.: Classical solutions and the Glassey-Strauss theorem for the 3D Vlasov-Maxwell system, *Arch. Rational Mech. Anal.* **170**, 1-15 (2013)
- [2] DiPERNA R. & LIONS P.-L.: Global weak solutions of Vlasov-Maxwell systems, *Comm. Pure Appl. Math.* **42**, 729-757 (1989)
- [3] GLASSEY R.: *The Cauchy Problem in Kinetic Theory*, SIAM, Philadelphia 1996
- [4] GLASSEY R. & SCHAEFFER J.: Global existence for the relativistic Vlasov-Maxwell system with nearly neutral initial data, *Comm. Math. Phys.* **119**, 353-384 (1988)

- [5] GLASSEY R. & SCHAEFFER J.: The “two and one-half-dimensional” relativistic Vlasov Maxwell system, *Comm. Math. Phys.* **185**, 257-284 (1997)
- [6] GLASSEY R. & STRAUSS W.: Remarks on collisionless plasmas, in: *Fluids and Plasmas: Geometry and Dynamics (Boulder, Colo., 1983)*, Contemp. Math. Vol. 28, AMS, Providence/Rhode Island 1984, pp. 269-279
- [7] GLASSEY R. & STRAUSS W.: Singularity formation in a collisionless plasma could occur only at high velocities, *Arch. Rational Mech. Anal.* **92**, 59-90 (1986)
- [8] GLASSEY R. & STRAUSS W.: Absence of shocks in an initially dilute collisionless plasma, *Comm. Math. Phys.* **113**, 191-208 (1987)
- [9] GLASSEY R. & STRAUSS W.: Large velocities in the relativistic Vlasov-Maxwell equations, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **36**, 615-627 (1989)
- [10] HORST E.: Symmetric plasmas and their decay, *Comm. Math. Phys.* **126**, 613-633 (1990)
- [11] KLAINERMAN S. & STAFFILANI G.: A new approach to study the Vlasov-Maxwell system, *Comm. Pure Appl. Anal.* **1**, 103-125 (2002)
- [12] KUNZE M.: *Das relativistische Vlasov-Maxwell-System partieller Differentialgleichungen zu Anfangsdaten mit nichtkompaktem Träger*, Diploma thesis, LMU München 1991
- [13] LUK J. & STRAIN R.: A new continuation criterion for the relativistic Vlasov-Maxwell system, preprint [arXiv:1406.0165v1](https://arxiv.org/abs/1406.0165v1)
- [14] LUK J. & STRAIN R.: Strichartz estimates and moment bounds for the relativistic Vlasov-maxwell system II. Continuation criteria in the 3D case, preprint [arXiv:1406.0169v1](https://arxiv.org/abs/1406.0169v1)
- [15] PALLARD CH.: On the boundedness of the momentum support of solutions to the relativistic Vlasov-Maxwell system, *Indiana Univ. Math. J.* **54**, 1395-1409 (2005)
- [16] REIN G.: Generic global solutions of the relativistic Vlasov-Maxwell system of plasma physics, *Comm. Math. Phys.* **135**, 41-78 (1990)
- [17] SCHAEFFER J.: The classical limit of the relativistic Vlasov-Maxwell system, *Comm. Math. Phys.* **104**, 403-421 (1986)
- [18] SOGGE C.D.: *Lectures on Nonlinear Wave Equations*, International Press, Boston 1995
- [19] SOSPEDRA-ALFONSO R. & ILLNER R.: Classical solvability of the relativistic Vlasov-Maxwell system with bounded spatial density, *Math. Methods Appl. Sci.* **33**, 751-757 (2010)
- [20] STRAUSS W.A.: *Nonlinear Wave Equations*, Conference Board of the Mathematical Sciences/Regional Conference Series in Mathematics No. 73, AMS, Providence/Rhode Island 1989